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COMPUTATION OF BESSEL AND NEUMANN FUNCTIONS.(U)

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Contract NObsr-85185
Index No. SF001-03-01, Task 8100

TRACOR

TECHNICAL MEMORANDUM

COMPUTATION OF BESSEL AND NEUMANN FUNCTIONS

Prepared for

The Bureau of Ships
Code 689B

December 11, 1962

T R A C O R, INC.

1701 Guadalupe St. Austin 1, Texas GR6-6601

12

TRACOR, INC. 1701 Guadalupe St. Austin 1, Texas

(15) Contract NObsr-85185

Index No. SF001-03-01, Task 8100

(9) TECHNICAL ~~MEMORANDUM~~ memo,

(6) COMPUTATION OF BESSEL AND NEUMANN FUNCTIONS.

(11) 13 Dec 62

(12) 14p

Prepared for

The Bureau of Ships
Code 689B

(16) F00103 (17) SF0010301

This technical memorandum contains partial results obtained during an analytical study of the sound field near a dome-baffle-transducer complex.

December 13, 1962

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COMPUTATION OF BESSEL AND NEUMANN FUNCTIONS

This memorandum describes the method by which a program was developed for evaluating Bessel and Neumann functions of integral order with real positive arguments. This method was developed specifically for use on small scale computers for which high computing speed can be obtained if excessive precision is avoided. The program allows the functions to be computed to five significant digits using six digits in the computation.

The work described in this memorandum was carried out to allow computation of numerical values for solutions to the wave equation in cylindrical coordinates. This problem arose in connection with a program of work in progress for evaluating the sound pressure field near a baffle-transducer complex.

Three different methods are used to evaluate the functions depending on the value of the order and the argument. The range of the order and argument for each of the three methods is shown in Figure 1. Note that the three regions completely cover all positive integral values of order and all positive values of the argument. The reasons for dividing the N-X plane into the regions shown in Figure 1 will be discussed below.

I. BESSEL FUNCTIONS

A. Region 1

The solution of Bessel's differential equation of the first kind is usually referred to as the Bessel function. This function may be expressed as a power series,

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{n+2m}}{m!\,(n+m)!} = \sum_{m=0}^{\infty} T_m \qquad (1)$$

where

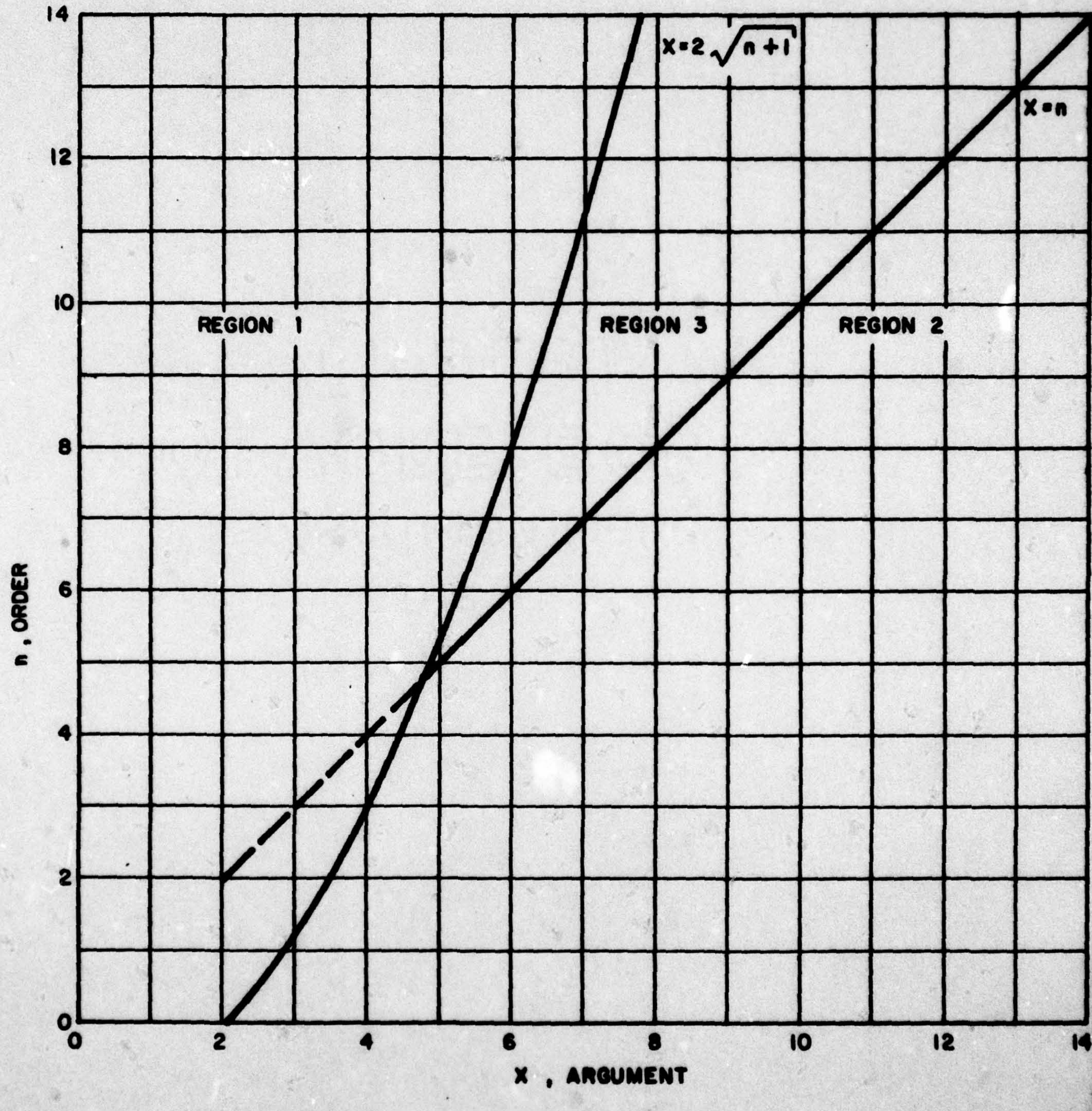

Fig. 1

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$$T_m = \frac{(-1)^m \left(\frac{x}{2}\right)^{n+2m}}{m!\ (n+m)!} = -T_{m-1} \frac{\left(\frac{x}{2}\right)^2}{m(n+m)} \qquad (2)$$

and

$$T_o = \frac{\left(\frac{x}{2}\right)^n}{n!} \qquad (3)$$

$J_n(x)$ is the Bessel function of order n and argument x. The discussion in this report will be restricted to non-negative integral values of n and real non-negative values of x. The power series for the Bessel function is convergent for any integral order and any argument. The usefulness of the power series as a computational algorithm depends on the behavior of the terms of the sum.

Since the terms of the sum alternate in sign, one must consider the possibility of loss of significance due to subtraction. To examine the sum further, consider the computation of $J_o(x)$,

$$T_m = \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{(m!)^2} \qquad n = 0 \qquad (4)$$

The steps in evaluating $J_o(x)$ for x = 0, 1, 2, 6, and 10 are shown in Table 1. From Table 1 it can be seen that the power series converges in a few terms only if the argument is small. If the argument is large the absolute value of successive terms diverges until the term number, m, exceeds $\frac{1}{2}\left[\sqrt{n^2 + x^2} - n\right]$. As can be seen from the computation of $J_o(10)$ the divergent characteristics of the sum leads to a subtractive loss of significance. The insignificant digits of the sums are circled. Note that only three significant digits are obtained in the computation of $J_o(10)$ using six digits of significance in the calculations. Hence, we

TABLE 1

SEQUENTIAL TERMS AND PARTIAL SUMS FOR $J_o(x)$

	x = 0		x = 1		x = 2		x = 6		x = 10	
m	T_m	S_m	T_m	S_m	T_m	S_m	T_m	S_m	T_m	S_m
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.0	1.00000	-.250000	.750000	-1.00000	.000000	-9.00000	-8.00000	-25.0000	-24.0000
2			.015625	.765625	.250000	.250000	20.2500	12.2500	156.250	132.250
3			-.000434	.765191	-.027777	.222222	-20.2500	-8.00000	-434.028	-301.778
4			.000005	.765196	.001736	.223958	11.3906	3.39060	678.169	376.391
5					-.000069	.223889	-4.10062	-.710020	-678.169	-301.778
6					.000002	.223891	1.02516	.315140	470.951	169.173
7							-.188295	.126845	-240.281	-71.1080
8							.026479	.153324	93.8598	22.7518
9							-.002942	.150382	-28.9691	-6.21730
10							.000265	.150647	7.24228	1.02498
11							-.000020	.150627	-1.49634	-.471360
12							.000001	.150628	.259781	-.211579
13									-.038429	-.250008
14									.004902	-.245106
15									-.000545	-.245651
Correct Value	1.00000			.765198		.223891		.150645		-.245936

see that the power series is a useful computational algorithm only if the argument is small when compared to the order.

To obtain a quantitative expression for the largest value of the argument which may be considered "small," consider equation (2) for m = 1,

$$\frac{T_1}{T_0} = - \frac{\left(-\frac{x}{2}\right)^2}{(n+1)} . \tag{5}$$

If the ratio of T_1 to T_o is equated to -1, the resulting expression can be solved to determine the value of the argument for which $T_1 = -T_o$,

$$x_1 = 2\sqrt{n + 1} . \tag{6}$$

If the argument is less than or equal to x_1, the power series may be used to compute the Bessel function since the absolute value of successive terms is non-increasing. If k digits of significance are used in computing $J_n(x)$, then the error due to subtraction is approximately $(\frac{x}{2})^n \frac{1}{n!} \cdot 10^{-k}$.

B. Region 2

1. General Method

On Figure 1 regions 2 and 3 are separated by the line of equal order and argument. For values of order greater than two the Bessel function and its derivative can be evaluated for equal order and argument using[1].

[1] G. N. Watson, A Treatise on the Theory of Bessel Functions, (Cambridge, 1958), 746.

$$J_n(n) = \frac{.447307}{n^{1/3}}\left[1 - \frac{1}{225n^2}\right] - \frac{.00586929}{n^{5/3}}\left[1 - \frac{1213}{14625n^2}\right], \quad (7)$$

$$J'_n(n) = \frac{.410850}{n^{2/3}}\left[1 + \frac{23}{3150n^2}\right] - \frac{.0894615}{n^{4/3}}\left[1 - \frac{947}{69300n^2}\right]. \quad (8)$$

If J_n and J'_n are known, then J_{n-1} and J'_{n-1} can be computed by the recursion relationships,

$$J_{n-1}(x) = J'_n(x) + \frac{n}{x} J_n(x), \quad (9)$$

$$J'_{n-1}(x) = \frac{n-1}{x} J_{n-1}(x) - J_n(x) \quad (10)$$

In theory if one can find the Bessel function and its derivative for any order, then the function can be computed for any order by successive application of Equations (9) and (10).

In practice one must exercise some caution in successive application of recursion formulae as subtractive loss of significance can occur. The loss of significance is not severe if the formulae are applied in region 2. However, severe significance loss does occur if these formulae are applied with increasing order in region 3.

To compute a Bessel function, $J_n(x)$, in region 2 the following steps are carried out:

a. The argument x is rounded to the nearest integer, x_o.
b. $J_{x_o}(x_o)$ and $J'_{x_o}(x_o)$ are computed using equations (7) and (8)

c. $J_n(x_o)$ and $J_n'(x_o)$ are computed by successive application of the recursion formulae.

d. $J_n(x)$ is computed using a Taylor's series expansion about the point x_o. $J_n(x_o)$ and $J_n'(x_o)$ are used to compute the coefficients of the expansion.[2]

The equations for the Taylor's series expansion are included here for completeness. However, the derivation of the equations and proof of convergence are not included here.

2. Taylor's Series Expansion

The solutions to Bessel's equation, $S_n(x)$, may be expressed as a Taylor's series expansion about the point x_o,

$$S_n(x) = \sum_{m=0}^{\infty} A_m \Delta^m, \tag{11}$$

where

$$\Delta = x - x_o, \tag{12}$$

$$A_o = S_n(x_o), \tag{13}$$

$$A_1 = S_n'(x_o), \tag{14}$$

$$A_m = -\frac{1}{m(m-1)x_o^2}\Big\{(2m-3)(m-1)x_oA_{m-1} + [(m-2)^2 + x_o^2 - n^2]A_{m-2} + 2x_oA_{m-3} + A_{m-4}\Big\}. \quad m \geq 2 \tag{15}$$

[2] Technical Memorandum (TRACOR), "The Evaluation of Bessel and Neumann Functions Through the Use of Taylor's Series Solutions to Bessel's Differential Equations."

and

$$A_{-1} = A_{-2} = 0. \tag{16}$$

By the proper choice of the coefficients A_o and A_1 any solution of Bessel's equation can be generated. Specifically if,

$$A_o = J_n(x_o)$$

and

$$A_1 = J_n'(x_o)$$

then the Bessel function will be generated. Because x_o is chosen by rounding x to the nearest integer $|\Delta| \leq .5$, the solution will therefore converge in a few terms. The convergence rate will be discussed in more detail below.

C. Region 3

1. General Method

It was noted earlier that successive application of the recursion formulae in region 3 with increasing order leads to severe subtractive loss of significance. Hence, the basic method which was applied in region 2 will not yield satisfactory results in region 3.

Earlier it was shown that in region 1 the Bessel function can be evaluated using the power series expansion about the origin shown in equation (1). Differentiation of equation (1) yields

$$J_n'(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (n + 2m) \left(\frac{x}{2}\right)^{n+2m-1}}{2(m!)(n + m)!} \tag{17}$$

To compute a Bessel function, $J_n(x)$, in region 3 the following steps are carried out:

a. Compute $x_o = 2\sqrt{n + 1}$. In region 3, x_o is less than x.

b. Compute $J_n(x_o)$ using Equation (1).

c. Compute $J_n'(x_o)$ using Equation (17).

d. Using the Taylor's series expansion about the point x_o compute $J_n(x_o')$ and $J_n'(x_o')$ where x_o' is either x or the largest value of the argument which may be used without severe subtractive loss of significance in evaluating the Taylor's series expansion.

e. If it is not possible to compute $J_n(x)$ in step d, the values computed at x_o' are used to repeat step d until it is possible to compute $J_n(x)$.

Differentiation of Equation (11) yields the expression for $J_n'(x)$ needed in step d above.

$$S_n'(x) = \sum_{m=1}^{\infty} m \cdot A_m \Delta^{m-1} \tag{18}$$

Equations (11) and (18) converge in a few terms if Δ is small. However, if Δ is large, Equations (11) and (18) display slow convergence and subtractive loss of significance. To avoid this, one must develop some method of choosing an upper limit for Δ. Two criteria must be satisfied.

$$|\Delta| \leq (3x_o)^{1/3} \tag{19}$$

and

$$|\Delta| \leq \sqrt{\left|\frac{2x_o^2}{x_o^2 - n^2}\right|}. \tag{20}$$

If Equations (19) and (20) are both satisfied, then Equations (11) and (18) will converge in a few terms without severe subtractive loss of significance.*

II. NEUMANN FUNCTIONS

A. Region 1

The solution of Bessel's differential equation of the second kind is usually referred to as the Neumann function. This function may be expressed as a series expansion,

$$N_n(x) = 2[\gamma + \ln(\frac{x}{2})J_n(x) - \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} (\frac{x}{2})^{2m-n}$$

$$- \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{x}{2})^{n+2m}}{m!(n+m)!} \left[\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{m} + \frac{1}{1} + \frac{1}{2}\right.$$

$$\left. + \cdots + \frac{1}{n+m}\right] \tag{21}$$

where $N_n(x)$ = The Neumann function of order n and argument x;

γ = Euler's constant ≅ .5772157.

Equation (21) may be used to evaluate the Neumann function in region 1. The convergence characteristics and

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*Equations (19) and (20) are derived from the recursion Equation (15) and should be regarded only as a sufficient condition, not a necessary condition.

subtractive loss of significance are very similar to those discussed earlier for the Bessel function.

B. Regions 2 and 3

The general method used to compute Neumann functions in regions 2 and 3 is the same as the method used to compute Bessel functions in region 2.

For values of order greater than two the Neumann function and its derivative can be evaluated for equal order and argument as,

$$N_n(n) = -\frac{.774759}{n^{1/3}}\left[1 - \frac{1}{225n^2}\right] - \frac{.0101659}{n^{5/3}}\left[1 - \frac{1213}{14625n^2}\right] \tag{22}$$

$$N_n'(n) = \frac{.711613}{n^{2/3}}\left[1 + \frac{23}{3150n^2}\right] + \frac{.154952}{n^{4/3}}\left[1 - \frac{947}{69300n^2}\right]. \tag{23}$$

The recursion formulae for the Neumann functions are identical to those for the Bessel function, so equations (9) and (10) may be restated as,

$$N_{n-1}(x) = N_n'(x) + \frac{n}{x} N_n(x), \tag{24}$$

$$N_{n-1}'(x) = \frac{n-1}{x} N_{n-1}(x) - N_n(x). \tag{25}$$

Successive application of Equations (24) and (25) allows one to compute N and N' for any integral order. Severe subtractive loss of significance does not occur when the recursion formulae are applied to the Neumann function in region 2 with decreasing order, or in region 3 with increasing order.

To compute a Neumann function, $N_n(x)$ in region 2 or 3 the following steps are carried out.

1. The argument is rounded to the nearest integer, x_o,
2. $N_{x_o}(x_o)$ and $N'_{x_o}(x_o)$ are computed using Equations (22) and (23).
3. $N_n(x_o)$ and $N'_n(x_o)$ are computed by successive application of the recurrence formulae
4. $N_n(x)$ is computed using the Taylor's series expansion about to the point x_o.

III. CONCLUSION

The method developed in this report allows one to compute the Bessel and Neumann functions of integral order with real positive arguments. Approximately five digits of accuracy are obtained using six digits in the computation. Subtractive loss of significance has been minimized in the method developed to avoid time consuming multiple precision arithmetic operations which would otherwise be required on a small scale computer. This allows one to apply a small scale computer to an entire class of problems which were previously restricted to solution on a large scale machine.